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Projective Modules of Finite Type over the Supersphere $S^{2,2}$

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Abstract

In the spirit of noncommutative geometry we construct all inequivalent vector bundles over the $(2,2)$-dimensional supersphere $S^{2,2}$ by means of global projectors p via equivariant maps. Each projector determines the projective module of finite type of sections of the corresponding 'rank 1' supervector bundle over $S^{2,2}$. The canonical connection $\nabla = p \circ d$ is used to compute the Chern numbers by means of the Berezin integral on $S^{2,2}$. The associated connection 1-forms are graded extensions of monopoles with not trivial topological charge. Supertransposed projectors gives opposite values for the charges. We also comment on the K-theory of $S^{2,2}$.

This work is dedicated to Anna

1 Preliminaries and Introduction

The Serre-Swan's theorem [23, 8] constructs a complete equivalence between the category of (smooth) vector bundles over a (smooth) compact manifold M and bundle maps, and the category of finite projective modules over the commutative algebra $C(M)$ of (smooth) functions over M and module morphisms. The space $\Gamma(M, E)$ of smooth sections of a vector bundle $E \to M$ over a compact manifold M is a finite projective module over the commutative algebra $C(M)$ and every finite projective $C(M)$-module can be realized as the module of sections of some vector bundle over M.

In the context of noncommutative geometry [7], where a noncommutative algebra $\mathcal{A}$ is the analogue of the algebra of smooth functions on some 'virtual noncommutative space', finite projective (left/right) modules over $\mathcal{A}$ have been used as algebraic substitutes for vector bundles, notably in order to construct noncommutative gauge and gravity theories (see for instance, [6, 7, 9, 13, 19]). In fact, in noncommutative geometry there seems to be more that one possibility for the analogue of the category of vector bundles [14].

On the other hand, there is a generalization of ordinary geometry which loosely speaking goes under the name of *supergeometry*. Supergeometry can hardly be considered noncommutative geometry and, indeed, one usually labels it *graded commutative geometry*. In this paper we present a finite-projective-module description of all not trivial monopoles configurations on the $(2,2)$-dimensional supersphere $S^{2,2}$. This will be done by constructing a suitable global projector p in the graded matrix algebra $\mathbb{M}_{|n|,|n|+1}(G(S^{2,2}))$, n being the value of the topological charge, while $G(S^{2,2})$ denotes the graded algebra of superfunctions on $S^{2,2}$. In the spirit of Serre-Swan theorem, the projector p determines the $G(S^{2,2})$-module $\mathcal{E}$ of sections of the supervector bundles on which monopoles live, as its image in the trivial module $G(S^2)^{2|n|+1}$ (corresponding to the trivial rank $(2|n|+1)$ supervector bundle over $S^{2,2}$), i.e. $\mathcal{E} = p(G(S^2)^{2|n|+1})$. The value of the topological charge is computed by taking the Berezin integral on $S^{2,2}$ of a suitable form. These monopoles will be also called Grassmann (or graded) monopoles. A description of a Grassmann monopole on a supersphere, as a strong connection in the framework of the theory of Hopf-Galois extensions, is in [11].

We refer to [15] for a friendly approach to modules of several kind (including finite projective). Throughout the paper we shall avoid writing explicitly the exterior product symbol for forms.

2 A Few Elements of Graded Algebra and Geometry

For us, graded will be synonymous of $\mathbb{Z}_2$-graded with the grading denoted as follows. If $M = M_0 \oplus M_1$, then $|m| = j$ means $m \in M_j$. The element m is said to be homogeneous if either $m \in M_0$ or $m \in M_1$. Elements of M_0 (resp. of M_1) are called *even* (resp. *odd*). A morphism $\phi : M \to N$ of graded structures is said to be *even* [resp. *odd*] if $\phi(M_j) \subseteq N_j$ [resp. $\phi(M_j) \subseteq N_{j+1}$, mod. 2].

With $B_L = (B_L)_0 + (B_L)_1$ we shall indicate a real Grassmann algebra with L generators. For simplicity we shall assume that $L < \infty$; mild assumptions (on the linear span of the products of odd elements) would allow to treat the case $L = \infty$ as well. Here B_L

is a graded commutative algebra, namely,

$$ab \in (B_L)_{|a|+|b|} \ , \quad ab = (-1)^{|a||b|} ba \ , \tag{2.1}$$

if the elements $a, b \in B_L$ are homogeneous. The algebra B_L can also be written as $B_L = \mathbb{R} \oplus N_L$ with N_L the nilpotent ideal. There are natural projections $\sigma : B_L \to \mathbb{R}$ and $s : B_L \to N_L$ which are called the *body* and the *soul* maps respectively. The Cartesian product ${B_L}^{m+n}$ is made into a graded B_L-module by setting

$$\begin{aligned} &{B_L}^{m+n} = {B_L}^{m,n} \ \oplus \ {B_L}^{\overline{m},\overline{n}} \ , \\ &{B_L}^{m,n} =: (B_L)_0{}^m \times (B_L)_1{}^n \ , \quad {B_L}^{\overline{m},\overline{n}} =: (B_L)_1{}^m \times (B_L)_0{}^n \ . \end{aligned} \tag{2.2}$$

The (m,n)-dimensional *superspace* ${B_L}^{m,n}$ is naturally a $(B_L)_0$-module. If L is finite, for consistency one must assume that $n \leq L$. A body map $\sigma^{m,n} : {B_L}^{m,n} \to \mathbb{R}^m$ is defined by

$$\sigma^{m,n}(x^1, \ldots, x^m \ ; \ y^1, \ldots, y^n) = \Big(\sigma(x^1), \ldots, \sigma(x^m)\Big) \ . \tag{2.3}$$

This map is used to endow ${B_L}^{m,n}$ with a topology, called *De Witt topology*, whose open sets are the inverse images of open sets in $\mathbb{R}^m$ through $\sigma^{m,n}$.

A graded B_L-module M is said to be *free of dimension* (m,n) if it is free of rank $m+n$ over B_L and has a basis formed by m even and n odd elements. The module M is said to be *projective (of finite type)* if it is the direct summand of a free module (of finite dimensionality). Any right graded B_L-module M can be turned into a left module, and viceversa, by defining

$$am =: (-1)^{|m||a|} ma \ , \quad \forall \ a \in B_L \ , \ m \in M \ , \tag{2.4}$$

which are homogeneous. Due to this fact, we shall not pay attention in distinguishing between right and left structures.

The collection $\mathbb{M}_{m+n}(B_L)$ of $(m+n)\times(m+n)$ matrices with entry in B_L is a graded B_L-module of dimension $(m^2+n^2, 2mn)$. It is given a grading in such a manner that its even part, denoted by $\mathbb{M}_{m,n}(B_L)$, is made of matrices of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ . \tag{2.5}$$

Here A and D are $m \times m$ and $n \times n$ matrices respectively, both with entries in $(B_L)_0$, whereas B and C are $m \times n$ and $n \times m$ matrices respectively, both with entries in $(B_L)_1$. Odd matrices in $\mathbb{M}_{m+n}(B_L)$ have the same form as (2.5) but now A and D have entries in $(B_L)_1$, whereas B and C have entries in $(B_L)_0$.
The B_L-module $\mathbb{M}_{m+n}(B_L)$ is made a graded Lie B_L algebra (also, a Lie superalgebra over B_L) by endowing it with a graded bracket whose definition on homogeneous elements is

$$[X, Y] =: XY - (-1)^{|X||Y|} YX \ . \tag{2.6}$$

The bracket is graded antisymmetric and satisfy a graded Jacobi identity. The *super-transpose* of the element (2.5) in $\mathbb{M}_{m+n}(B_L)$ is defined as,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} = \begin{pmatrix} A^t & (-1)^{|X|} \ C^t \\ -(-1)^{|X|} \ B^t & D^t \end{pmatrix} \ , \tag{2.7}$$

with the superscript ^{t} denoting usual matrix transposition. Then, one has

$$(XY)^{st} = (-1)^{|X||Y|}\; Y^{st} X^{st} \; . \tag{2.8}$$

In the present context, the ordinary trace tr is replaced by the *supertrace* Str which, for a matrix of the form (2.5) is defined as follows,

$$Str(X) =: tr(A) - (-1)^{|X|} tr(D) \; . \tag{2.9}$$

The supertrace obeys graded versions of the usual properties of a trace. In particular,

$$\begin{aligned} &Str(X^{st}) = Str(X) \\ &Str(XY) = (-1)^{|X||Y|} Str(YX) \; . \end{aligned} \tag{2.10}$$

The collection $GL_{m,n}(B_L)$ of invertible matrices in $\mathbb{M}_{m,n}(B_L)$ is naturally a super Lie group. If H is any matrix in $GL_{m,n}(B_L)$, one has that

$$Str(HXH^{-1}) = Str(X) \; . \tag{2.11}$$

On elements of $GL_{m,n}(B_L)$ one defines a *superdeterminant* (or *Berezinian*) which is valued in $(B_L)_0^*$, the group of invertible elements of $(B_L)_0$ [3]. First of all, one proves that if the matrix X has the form (2.5), then X is invertible if and only if A and D are invertible as ordinary matrices with entries in $(B_L)_0$. Then, if $X \in GL_{m,n}(B_L)$, its superdeterminant is defined as

$$Sdet(X) =: det(A - BD^{-1}C)\; det(D^{-1}) \; . \tag{2.12}$$

Again, the superdeterminant obeys graded versions of the usual properties of the determinant [3, 21],

$$\begin{aligned} &Sdet(X^{st}) = Sdet(X) \\ &Sdet(XY) = (-1)^{|X||Y|} Sdet(X) Sdet(Y) \; , \end{aligned} \tag{2.13}$$

for all $X, Y \in GL_{m,n}(B_L)$

All previous considerations and definitions concerning modules and matrices can be extended to any graded commutative algebra. In particular, we shall consider graded projective modules of finite type over graded commutative algebras of superfunctions and matrices with entries in graded commutative algebras of superfunctions.

In this paper we shall not dwell upon the different definitions of superstructures while referring, for instance, to [1]. Indeed, the only supermanifolds we shall consider are the so called *De Witt supermanifolds* [12]. One says that a (m, n)-dimensional supermanifold S is De Witt if it is locally modeled on ${B_L}^{m,n}$ and has an atlas such that the images of the coordinate maps are open in the De Witt topology of ${B_L}^{m,n}$. We shall denote by $G^\infty(S, B_L)$ the graded B_L-algebra of B_L-valued supersmooth functions on the supermanifold S. In a coordinate neighborhood, elements of $G^\infty(S, B_L)$ have a usual superfield expansion in the odd coordinate functions. Finally, we mention that it has been shown [22] that a De Witt (m, n) supermanifold S is a locally trivial fibre bundle over an ordinary m-dimensional manifold S_0, with a vector fibre. The manifold S_0 is called the body of S and the bundle projection $\Phi : S \to S_0$ is given in local bundle coordinates by the body map $\sigma^{m,n}$.

Let $C_L =: B_L \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of B_L. A *complex super line bundle* over a supermanifold can be thought of either as a rank $(1,0)$ or a rank $(0,1)$ super vector bundle since in both cases the standard fiber is C_L while the structure group is $(C_L)_0^* \simeq GL_{1,0}(C_L) \simeq GL_{0,1}(C_L)$, the group of invertible even elements in C_L. For this reason we shall not distinguish between these two cases and refer to the final Section for additional remarks. In the spirit of the Serre-Swan theorem, supervector bundles over De Witt supermanifolds will be 'identified' with (finite) graded projective modules of sections over the algebra of superfunctions over the base supermanifold. This is due to the fact that, contrary to what happens for a general supermanifold, any super vector bundle over a De Witt supermanifold admits a connection [1]. By the arguments in [10] the existence of a connection is equivalent to the module of sections being projective.

Finally, we remind that, again in contrast with what happens for a general supermanifold, complex super line bundles over a De Witt supermanifold are classified by their *obstruction class* and so they are in bijective correspondence with elements in the integer sheaf cohomology group $\check{H}^2(M,\mathbb{Z})$. For a line bundle, the obstruction class is essentially the *first Chern class* of the bundle. By using the morphism $j : \check{H}^2(M,\mathbb{Z}) \to H^2_{SDR}(M)$, the latter being the de Rham cohomology group of superforms, the obstruction class of complex super line bundles over a De Witt supermanifolds M can be realized as a super de Rham cohomology class of M [1]. A representative for this class can be given in term of the curvature of a connection on the bundle, the choice of the connection being immaterial since different connections yield the same cohomology class. We shall represent the Chern class of a complex line superbundle by means of the curvature of a canonical connection which, is a sense, it is determined by the bundle itself.

3 The Hopf Fibration over the Supersphere $S^{2,2}$

3.1 The Supergroup $UOSP(1,2)$

We shall describe the basic facts about the Lie supergroup $UOSP(1,2)$ that we need in this paper while referring to [4] for additional details.

Let $osp(1,2)$ be the Lie B_L superalgebra of dimension $(3,2)$ with even generators A_0, A_1, A_2 and odd generator R_+, R_-, explicitly given in matrix representation by

$$A_0 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_1 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$
$$R_+ = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R_- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

Thus, a generic element $X \in osp(1,2)$ is written as $X = \sum_{k=0,1,2} a_k A_k + \sum_{\alpha=+,-} \eta_\alpha R_\alpha$ with $a_k \in (B_L)_0$, $\eta_\alpha \in (B_L)_1$. The basis elements (3.1) are closed under graded commutator. In particular, the three even elements A_0, A_1, A_2 generate the Lie algebra $so(2) \simeq su(2)$.

If the integer L is taken to be even, on the complexification $C_L = B_L \otimes_{\mathbb{R}} \mathbb{C}$ there exists

[21] an even graded involution

$$\begin{array}{l} ^\diamond \ : \ C_L \ \rightarrow \ C_L \ , \\ \qquad |x^\diamond| = |x| \ , \quad \forall \ x \in (C_L)_{|x|} \ , \qquad (cx)^\diamond = \overline{c} x^\diamond \ , \quad \forall \ c \in \mathbb{C} \ , \ x \in C_L \ , \end{array} \tag{3.2}$$

which in addition verifies the properties

$$(xy)^\diamond = x^\diamond \ y^\diamond \ , \quad \forall \ x, y \in C_L \ , \qquad x^{\diamond\diamond} = (-1)^{|x|} x \ , \quad \forall \ x \in (C_L)_{|x|} \ . \tag{3.3}$$

The superalgebra $uosp(1,2)$ is defined to be the 'real' subalgebra made of elements of the form

$$X = \sum_{k=0,1,2} a_k A_k + \eta R_+ + \eta^\diamond R_- \ , \quad a_k \in (C_L)_0 \ , \ a_k^\diamond = a_k \ , \quad \eta \in (C_L)_1 \ . \tag{3.4}$$

Indeed, one introduces an adjoint operation † which is defined on the bases (3.1) as

$$A_i^\dagger = -A_i \ , \quad i = 0,1,2 \ ; \quad R_+^\dagger = -R_- \ , \ R_-^\dagger = R_+ \ , \tag{3.5}$$

and is extended to the whole of $C_L \otimes_{\mathbb{R}} osp(1,2)$ by using the involution $^\diamond$. Then, the superalgebra $uosp(1,2)$ is identified as the collection of 'anti-hermitian' elements

$$uosp(1,2) = \{ X \in C_L \otimes_{\mathbb{R}} osp(1,2) \mid X^\dagger = -X \} \ . \tag{3.6}$$

The superalgebra $uosp(1,2)$ is the analogue of the compact real form of $C_L \otimes_{\mathbb{R}} osp(1,2)$.

Finally, the Lie supergroup $UOSP(1,2)$ is defined to be the exponential map of $uosp(1,2)$,

$$UOSP(1,2) =: \{ exp(X) \mid X \in uosp(1,2) \} \ . \tag{3.7}$$

A generic element $s \in UOSP(1,2)$ can be presented as the product of one-parameter subgroups,

$$\begin{array}{l} s = u\xi \ , \\ u = exp(a_0 A_0) exp(a_1 A_1) exp(a_1 A_1) \ , \quad a_k^\diamond = a_k \in (C_L)_0 \ , \\ \xi = exp(\eta R_+) exp(\eta^\diamond R_-) = exp(\eta R_+ + \eta^\diamond R_-) \ , \quad \eta \in (C_L)_1 \ , \end{array} \tag{3.8}$$

The last equality being a consequence of the nilpotency of the variable η. Explicitly, the element $s \in UOSP(1,2)$ can be parametrized as

$$s = \begin{pmatrix} 1 + \frac{1}{4}\eta\eta^\diamond & -\frac{1}{2}\eta & \frac{1}{2}\eta^\diamond \\ -\frac{1}{2}(a\eta^\diamond - b^\diamond\eta) & a(1 - \frac{1}{8}\eta\eta^\diamond) & -b^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) \\ -\frac{1}{2}(b\eta^\diamond + a^\diamond\eta) & b(1 - \frac{1}{8}\eta\eta^\diamond) & a^\diamond(1 - \frac{1}{8}\eta\eta^\diamond) \end{pmatrix} \ . \tag{3.9}$$

Here a, b and η are elements in the complex Grassmann algebra C_L with the restrictions $a_k^\diamond = a_k \in (C_L)_0$ and $\eta \in (C_L)_1$. Furthermore, the superdeterminant of the matrix (3.9) is constrained to be 1 and this yields the condition,

$$1 = Sdet(s) = aa^\diamond + bb^\diamond \ . \tag{3.10}$$

It may be worth stressing that (3.10) is a condition in the even part $(C_L)_0$ of the Grassmann algebra C_L and thus involves all even combinations of generators of the latter. By using (3.8) one also finds the adjoint of any element to be

$$s^\dagger =: \xi^\dagger u^\dagger \;=\; \begin{pmatrix} 1+\frac{1}{4}\eta\eta^\diamond & \frac{1}{2}(a^\diamond\eta+b\eta^\diamond) & \frac{1}{2}(b^\diamond\eta-a\eta^\diamond) \\ \frac{1}{2}\eta^\diamond & a^\diamond(1-\frac{1}{8}\eta\eta^\diamond) & b^\diamond(1-\frac{1}{8}\eta\eta^\diamond) \\ \frac{1}{2}\eta & -b(1-\frac{1}{8}\eta\eta^\diamond) & a(1-\frac{1}{8}\eta\eta^\diamond) \end{pmatrix}, \tag{3.11}$$

and checks that $ss^\dagger = s^\dagger s = 1$.

We shall also need $\mathcal{U}(1)$, the Grassmann extension of $U(1)$. It is realized as follows

$$\mathcal{U}(1) = \{w \in (C_L)_0 \mid ww^\diamond = 1\}. \tag{3.12}$$

By embedding $\mathcal{U}(1)$ in $UOSP(1,2)$ as

$$w \;\mapsto\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^\diamond \end{pmatrix}, \tag{3.13}$$

we may think of A_0 as the generator of $\mathcal{U}(1)$, i.e.

$$\mathcal{U}(1) \simeq \{exp(\lambda A_0) \mid \lambda \in (C_L)_0 \;,\; \lambda^\diamond = \lambda\}. \tag{3.14}$$

3.2 The Principal $\mathcal{U}(1)$ Bundle over $S^{2,2}$

To our knowledge, the $\mathcal{U}(1)$ principal fibration $\pi : UOSP(1,2) \to S^{2,2}$ over the $(2,2)$-dimensional supersphere was introduced for the first time in [18] and further studied in [2] were, in particular, it was shown that $S^{2,2}$ is a De Witt supermanifold over the usual sphere S^2. The fibration is explicitly realized as follows. The total space is the $(1,2)$-dimensional supergroup $UOSP(1,2)$ while the structure supergroup is $\mathcal{U}(1)$. We let $\mathcal{U}(1)$ act on the right on $UOSP(1,2)$. If we parametrize any $s \in UOSP(1,2)$ by $s = s(a,b,\eta)$, then this action can be represented as follows,

$$UOSP(1,2) \times \mathcal{U}(1) \;\to\; UOSP(1,2)\;, \quad (s,w) \mapsto s\cdot w = s(aw, bw, \eta w)\;. \tag{3.15}$$

This action leaves unchanged the superdeterminant

$$Sdet(s\cdot w) = aw(aw)^\diamond + bw(bw)^\diamond = aa^\diamond + bb^\diamond = 1\;. \tag{3.16}$$

The bundle projection

$$\begin{aligned} &\pi : UOSP(1,2) \;\to\; S^{2,2} =: UOSP(1,2)/\mathcal{U}(1)\;, \\ &\pi(a,b,\eta) =: (x_0, x_1, x_2, \xi_+, \xi_-) \end{aligned} \tag{3.17}$$

can be given as the (co)-adjoint orbit through A_0 of the action of $UOSP(1,2)$ on $uosp(1,2)$. With $s^\dagger$ the adjoint of s as given in (3.11), one has that

$$\pi(s) =: s(\frac{2}{i}A_0)s^\dagger =: \sum_{k=0,1,2} x_k(\frac{2}{i}A_k) + \sum_{\alpha=+,-} \xi_\alpha(2R_\alpha)\;. \tag{3.18}$$

Explicitly,

$$
\begin{aligned}
x_0 &= (aa^\diamond - bb^\diamond)(1-\frac{1}{4}\eta\eta^\diamond) = (-1+2aa^\diamond)(1-\frac{1}{4}\eta\eta^\diamond) = (1-2bb^\diamond)(1-\frac{1}{4}\eta\eta^\diamond)\ ,\\
x_1 &= (a\bar{b}+b\bar{a})(1-\frac{1}{4}\eta\eta^\diamond)\ ,\\
x_2 &= i(a\bar{b}-b\bar{a})(1-\frac{1}{4}\eta\eta^\diamond)\ ,\\
\xi_- &= -\frac{1}{2}(a\eta^\diamond+\eta b^\diamond)\ ,\\
\xi_+ &= \frac{1}{2}(\eta a^\diamond - b\eta^\diamond)\ .
\end{aligned}
\tag{3.19}
$$

One sees directly that the x_k's are even, $x_k \in (C_L)_0$, and 'real', $x_k^\diamond = x_k$, while the ξ_α are odd, $\xi_\alpha \in (C_L)_1$, and such that ${\xi_-}^\diamond = \xi_+$ (and ${\xi_+}^\diamond = -\xi_-$). In addition, one finds that

$$
\begin{aligned}
\sum_{\mu=0}^{2}(x_\mu)^2 + 2\xi_-\xi_+ &= (aa^\diamond+bb^\diamond)^2(1-\frac{1}{2}\eta\eta^\diamond) + \frac{1}{2}(aa^\diamond+bb^\diamond)\eta\eta^\diamond\\
&= 1\ .
\end{aligned}
\tag{3.20}
$$

Thus, the base space $S^{2,2}$ is a $(2,2)$-dimensional sphere in the superspace $B_L^{3,2}$. It turns out that $S^{2,2}$ is a De Witt supermanifold with *body* the usual sphere S^2 in $\mathbb{R}^3$ [2], a fact that we shall use later. The inversion of (3.19) gives the basic (C_L-valued) invariant functions on $UOSP(1,2)$. Firstly, notice that

$$
\frac{1}{4}\eta\eta^\diamond = \xi_-\xi_+\ . \tag{3.21}
$$

Furthermore,

$$
\begin{aligned}
aa^\diamond &= \frac{1}{2}\Big[1+x_0(1+\xi_-\xi_+)\Big]\ ,\\
bb^\diamond &= \frac{1}{2}\Big[1-x_0(1+\xi_-\xi_+)\Big]\ ,\\
ab^\diamond &= \frac{1}{2}(x_1-ix_2)(1+\xi_-\xi_+)\ ,\\
\eta a^\diamond &= -(x_1+ix_2)\xi_- + (1+x_0)\xi_+\ ,\\
\eta b^\diamond &= (x_1-ix_2)\xi_+ - (1-x_0)\xi_-\ ,
\end{aligned}
\tag{3.22}
$$

a generic invariant (polynomial) function on $UOSP(1,2)$ being any function of the previous variables.

We shall denote with $\mathcal{B}_{C_L} =: G^\infty(UOSP(1,2), C_L)$ the graded algebra of C_L-valued smooth superfunctions on the total space $UOSP(1,2)$, while $\mathcal{A}_{C_L} =: G^\infty(S^{2,2}, C_L)$ will be the graded algebra of C_L-valued smooth superfunctions on the base space $S^{2,2}$. In the following, we shall identify $\mathcal{A}_{C_L}$ with its image in $\mathcal{B}_{C_L}$ via pull-back.

4 The Equivariant Maps and the Projectors

4.1 The Equivariant Maps

Just as it happens for the group $U(1)$, irreducible representations of the supergroup group $\mathcal{U}(1)$ are labeled by an integer $n \in \mathbb{Z}$, any two representations associated with different integers being inequivalent. They can be explicitly given as left representations on C_L,

$$\rho_n \ : \ \mathcal{U}(1) \times C_L \ \to \ C_L \ , \quad (w, c) \mapsto \rho_n(w) \cdot c =: w^n c \ . \tag{4.1}$$

In order to construct the corresponding equivariant maps $\varphi : UOSP(1,2) \to C_L$ we shall distinguish between the two cases for which the integer n is negative or positive. From now on, we shall take the integer n to be always positive and consider the two cases corresponding to $\mp n$.

4.1.1 The Equivariant Maps for Negative Labels

Given any positive integer, $n \in \mathbb{N}$, the equivariant maps $\varphi_{-n} : UOSP(1,2) \to C_L$ corresponding to the representation of $\mathcal{U}(1)$ labelled by $-n$ are of the form

$$\varphi_{-n}(\eta, a, b) = \frac{1}{2}\ \eta \sum_{j=0}^{n-1} a^{n-1-j} b^j f_j + (1 - \frac{1}{8}\eta\eta^\diamond) \sum_{k=0}^{n} a^{n-k} b^k g_k \ , \tag{4.2}$$

with $f_j,\ j = 1, \ldots, n-1$ and $g_k,\ k = 1, \ldots, n$ any C_L-valued functions which are invariant under the right action of $\mathcal{U}(1)$ on $UOSP(1,2)$. The reason for the choice of the additional invariant factor $(1 - \frac{1}{8}\eta\eta^\diamond)$ will be given later. Indeed,

$$\begin{aligned} \varphi_{-n}((\eta, a, b)w) &= \frac{1}{2}\ (\eta w) \sum_{j=0}^{n-1} (aw)^{n-1-j} (bw)^j f_j + (1 - \frac{1}{8}\eta\eta^\diamond) \sum_{k=0}^{n} (aw)^{n-k} (bw)^k g_k \\ &= w^n \varphi_{-n}(\eta, a, b) \\ &= \rho_{-n}(w)^{-1} \varphi_{-n}(\eta, a, b) \ . \end{aligned} \tag{4.3}$$

The functions f_j , g_k's being invariant, we shall think of them as C_L-valued functions on the base space $S^{2,2}$, namely as elements of the graded algebra $\mathcal{A}_{C_L}$. The space $G^\infty_{(-n)}(UOSP(1,2), C_L)$ of equivariant maps is a right module over the (pull-back of) superfunctions $\mathcal{A}_{C_L}$.

4.1.2 The Equivariant Maps for Positive Labels

Given any positive integer, $n \in \mathbb{N}$, the equivariant maps $\varphi_n : UOSP(1,2) \to C_L$ corresponding to the representation of $\mathcal{U}(1)$ labelled by n are of the form

$$\varphi_n(\eta, a, b) = \frac{1}{2}\ \eta^\diamond \sum_{j=0}^{n-1} (a^\diamond)^{n-1-j} (b^\diamond)^j f_j + (1 - \frac{1}{8}\eta\eta^\diamond) \sum_{k=0}^{n} (a^\diamond)^{n-k} (b^\diamond)^k g_k \ , \tag{4.4}$$

with $f_j,\ j=1,\ldots,n-1$ and $g_k,\ k=1,\ldots,n$ any C_L-valued functions which are invariant under the right action of $\mathcal{U}(1)$ on $UOSP(1,2)$. Indeed,

$$\begin{aligned}\varphi_n((\eta,a,b)w) &= \frac{1}{2}\,(\eta w)^\diamond\sum_{j=0}^{n-1}((aw)^\diamond)^{n-1-j}((bw)^\diamond)^j f_j + (1-\frac{1}{8}\eta\eta^\diamond)\sum_{k=0}^{n}(a^\diamond w)^{n-k}(b^\diamond w)^k g_k \\ &= (w^\diamond)^n\varphi_{-n}(\eta,a,b) \\ &= \rho_n(w)^{-1}\varphi_n(\eta,a,b)\ ,\end{aligned} \tag{4.5}$$

where we have used the fact that $w^\diamond = w^{-1}$. As before, we shall think of the functions f_j , g_k's as elements of the graded algebra $\mathcal{A}_{C_L}$. And the space $G^\infty_{(n)}(UOSP(1,2),C_L)$ of equivariant maps will again be a right module over $\mathcal{A}_{C_L}$.

4.2 The Projectors and the Bundles

We are now ready to introduce the projectors. Again we shall take the integer n to be positive and keep separated the two cases corresponding to $\mp n$.

4.2.1 The Construction of the Projectors for Negative Labels

Given any positive integer, $n\in\mathbb{N}$, let us consider the supervector-valued function with $(n,n+1)$-components given by,

$$\begin{aligned}\langle\psi_{-n}| =: \Big(\frac{1}{2}\eta\left(a^n,\ldots,\ \sqrt{\tbinom{n-1}{k}}a^{n-1-k}b^k\ ,\ldots,b^{n-1}\right);\\ (1-\frac{1}{8}\eta\eta^\diamond)\left(a^n,\ldots,\ \sqrt{\tbinom{n}{k}}a^{n-k}b^k\ ,\ldots,b^n\right)\Big)\ ,\end{aligned} \tag{4.6}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are the binomial coefficients.
The supervector-valued function (4.6) is normalized,

$$\begin{aligned}\langle\psi_{-n}|\psi_{-n}\rangle &= \frac{1}{4}\eta\eta^\diamond\sum_{j=0}^{n-1}\tbinom{n-1}{j}a^{n-j-1}b^j(a^\diamond)^{n-j-1}(b^\diamond)^j \\ &\qquad + (1-\frac{1}{4}\eta\eta^\diamond)\sum_{k=0}^{n}\tbinom{n}{k}a^{n-k}b^k(a^\diamond)^{n-k}(b^\diamond)^k \\ &= \frac{1}{4}\eta\eta^\diamond(aa^\diamond+bb^\diamond)^{n-1} + (1-\frac{1}{4}\eta\eta^\diamond)(aa^\diamond+bb^\diamond)^n \\ &= 1\ .\end{aligned} \tag{4.7}$$

Then, we can construct a projector in $\mathbb{M}_{n,n+1}(\mathcal{A}_{C_L})$ by

$$p_{-n} =: |\psi_{-n}\rangle\,\langle\psi_{-n}|\ . \tag{4.8}$$

It is clear that p_{-n} is a projector,

$$\begin{aligned}&p_{-n}^2 =: |\psi_{-n}\rangle\,\langle\psi_{-n}|\psi_{-n}\rangle\,\langle\psi_{-n}| = |\psi_{-n}\rangle\,\langle\psi_{-n}| = p_{-n}\ ,\\ &p_{-n}^\dagger = p_{-n}\ .\end{aligned} \tag{4.9}$$

Moreover, it is of rank 1 because its supertrace is the constant superfunction 1,

$$Str(p_{-n}) = \langle \psi_{-n} | \psi_{-n} \rangle = 1 \ . \tag{4.10}$$

The $\mathcal{U}(1)$-action will transform the vector (4.6) multiplicatively,

$$\langle \psi_{-n} | \ \mapsto \ \langle (\psi_{-n})^w | = w^n \langle \psi_{-n} | \ , \quad \forall \ w \in \mathcal{U}(1) \ . \tag{4.11}$$

As a consequence the projector p_{-n} is invariant,

$$p_{-n} \ \mapsto \ (p_{-n})^w = |(\psi_{-n})^w\rangle \langle (\psi_{-n})^w| = |\psi_{-n}\rangle (w^\diamond)^n w^n \langle \psi_{-n}| = |\psi_{-n}\rangle \langle \psi_{-n}| = p_{-n} \tag{4.12}$$

(being $w^\diamond w = 1$), and its entries are functions on the base space $S^{2,2}$, that is they are elements of $\mathcal{A}_{C_L}$ as it should be. Thus, the right module of sections $\Gamma^\infty(S^{2,2}, E^{(-n)})$ of the associated bundle is identified with the image of p_{-n} in the trivial rank $(2n+1)$ module $(\mathcal{A}_{C_L})^{2n+1}$ and the module isomorphism between sections and equivariant maps is given by,

$$\begin{aligned}
&\Gamma^\infty(S^{2,2}, E^{(-n)}) \ \leftrightarrow \ G^\infty_{(-n)}(UOSP(1,2), C_L) \ , \\
&\sigma = p_{-n} \begin{pmatrix} f_0 \\ \vdots \\ g_n \end{pmatrix} \ \leftrightarrow \ \varphi_\sigma(a,b) = \langle \psi_{-n} | \begin{pmatrix} f_0 \\ \vdots \\ g_n \end{pmatrix} \\
&\varphi_\sigma(a,b) = \frac{1}{2} \ \eta \sum_{j=0}^{n-1} \sqrt{\tbinom{n-1}{j}} a^{n-1-j} b^j f_j + (1 - \frac{1}{8} \eta \eta^\diamond) \sum_{k=0}^{n} \sqrt{\tbinom{n}{k}} a^{n-k} b^k g_k \ ,
\end{aligned} \tag{4.13}$$

with $f_j, \ j = 1, \ldots, n-1$ and $g_k, \ k = 1, \ldots, n$ generic elements in $\mathcal{A}_{C_L}$. By comparison with (4.2) it is obvious that the previous map is a module isomorphism, the extra factors given by the binomial coefficients being inessential to this purpose, since they could be absorbed in a redefinition of the functions.

4.2.2 The Construction of the Projectors for Positive Labels

Given any positive integer, $n \in \mathbb{N}$, let us consider the supervector-valued function with $(n, n+1)$-components given by,

$$\begin{aligned}
\langle \psi_n | =: \Bigg(&\frac{1}{2} \eta^\diamond \Big((a^\diamond)^n, \ldots, \ \sqrt{\tbinom{n-1}{k}} (a^\diamond)^{n-1-k} (b^\diamond)^k \ , \ldots, (b^\diamond)^{n-1} \Big) \ ; \\
&(1 - \frac{1}{8} \eta \eta^\diamond) \Big((a^\diamond)^n, \ldots, \ \sqrt{\tbinom{n}{k}} (a^\diamond)^{n-k} (b^\diamond)^k \ , \ldots, (b^\diamond)^n \Big) \Bigg) \ .
\end{aligned} \tag{4.14}$$

The supervector-valued function (4.14) is normalized,

$$\begin{aligned}
\langle \psi_n | \psi_n \rangle \ &= \ -\frac{1}{4} \eta^\diamond \eta \sum_{j=0}^{n-1} \tbinom{n-1}{j} (a^\diamond)^{n-j-1} (b^\diamond)^j a^{n-j-1} b^j \\
&\qquad + (1 - \frac{1}{4} \eta \eta^\diamond) \sum_{k=0}^{n} \tbinom{n}{k} (a^\diamond)^{n-k} (b^\diamond)^k a^{n-k} b^k \\
&= \ \frac{1}{4} \eta \eta^\diamond (a a^\diamond + b b^\diamond)^{n-1} + (1 - \frac{1}{4} \eta \eta^\diamond)(a a^\diamond + b b^\diamond)^n \\
&= \ 1 \ .
\end{aligned} \tag{4.15}$$

Then, we can construct a projector in $\mathbb{M}_{n,n+1}(\mathcal{A}_{C_L})$ by

$$p_n =: |\psi_n\rangle \langle\psi_n| \ . \tag{4.16}$$

It is clear that p_n is a projector,

$$\begin{aligned} p_n^2 &=: |\psi_n\rangle \langle\psi_n|\psi_n\rangle \langle\psi_n| = |\psi_n\rangle \langle\psi_n| = p_n \ , \\ p_n^\dagger &= p_n \ . \end{aligned} \tag{4.17}$$

Moreover, it is of rank 1 because its supertrace is the constant superfunction 1,

$$Str(p_n) = \langle\psi_n|\psi_n\rangle = 1 \ . \tag{4.18}$$

The $\mathcal{U}(1)$-action will transform the vector (4.14) multiplicatively,

$$\langle\psi_n| \ \mapsto \ \langle(\psi_n)^w| = (w^\diamond)^n \langle\psi_n| \ , \quad \forall\ w \in \mathcal{U}(1) \ . \tag{4.19}$$

As a consequence the projector p_n is invariant,

$$p_n \ \mapsto \ (p_n)^w = |(\psi_n)^w\rangle \langle(\psi_n)^w| = |\psi_n\rangle\, w^n (w^\diamond)^n \langle\psi_n| = |\psi_n\rangle \langle\psi_n| = p_n \ , \tag{4.20}$$

and its entries are functions on the base space $S^{2,2}$, that is they are elements of $\mathcal{A}_{C_L}$ as it should be. Thus, the right module of sections $\Gamma^\infty(S^{2,2}, E^{(n)})$ of the associated bundle is identified with the image of p_n in the trivial rank $(2n+1)$ module $(\mathcal{A}_{C_L})^{2n+1}$ and the module isomorphism between sections and equivariant maps is given by,

$$\begin{aligned} &\Gamma^\infty(S^{2,2}, E^{(n)}) \ \leftrightarrow \ G^\infty_{(n)}(UOSP(1,2), C_L) \ , \\ &\sigma = p_n \begin{pmatrix} f_0 \\ \vdots \\ g_n \end{pmatrix} \ \leftrightarrow \ \varphi_\sigma(a,b) = \langle\psi_n| \begin{pmatrix} f_0 \\ \vdots \\ g_n \end{pmatrix} \\ &\varphi_\sigma(a,b) = \frac{1}{2}\, \eta^\diamond \sum_{j=0}^{n-1} \sqrt{\binom{n-1}{j}} (a^\diamond)^{n-1-j} (b^\diamond)^j f_j \\ &\qquad\qquad + (1 - \frac{1}{8}\eta\eta^\diamond) \sum_{k=0}^{n} \sqrt{\binom{n}{k}} (a^\diamond)^{n-k} (b^\diamond)^k g_k \ , \end{aligned} \tag{4.21}$$

with $f_j,\ j = 1, \ldots, n-1$ and $g_k,\ k = 1, \ldots, n$ generic elements in $\mathcal{A}_{C_L}$. By comparison with (4.4) it is obvious that the previous map is a module isomorphism, the extra factors given by the binomial coefficients being inessential to this purpose, since they could be absorbed in a redefinition of the functions.

The vector superfunctions $\langle\psi_n|$ and $\langle\psi_{-n}|$ are one the supertransposed of the other, that is,

$$\langle\psi_n| = (|\psi_{-n}\rangle)^{st} = \left\langle(\psi_{-n})^\diamond\right| \ , \tag{4.22}$$

and the corresponding projectors are related by supertransposition,

$$p_n = (p_{-n})^{st} \ . \tag{4.23}$$

Thus, by transposing a projector we get an inequivalent one (unless the projector is the identity).

Examples. Here we give the explicit projectors corresponding to the lowest values of the charges [16],

$$p_{-1} = \frac{1}{2}\begin{pmatrix} 2\xi_+\xi_- \ ; & (x_1+ix_2)\xi_- - (1+x_0)\xi_+ \ ; & -(x_1-ix_2)\xi_+ + (1-x_0)\xi_- \\ -(x_1-ix_2)\xi_+ - (1+x_0)\xi_- \ ; & 1+x_0+\xi_+\xi_- \ ; & x_1-ix_2 \\ -(x_1+ix_2)\xi_- - (1-x_0)\xi_+ \ ; & x_1+ix_2 \ ; & 1-x_0+\xi_+\xi_- \end{pmatrix}, \tag{4.24}$$

$$p_{1} = \frac{1}{2}\begin{pmatrix} 2\xi_+\xi_- \ ; & -(x_1-ix_2)\xi_+ - (1+x_0)\xi_- \ ; & -(x_1+ix_2)\xi_- - (1-x_0)\xi_+ \\ -(x_1+ix_2)\xi_- + (1+x_0)\xi_+ \ ; & 1+x_0+\xi_+\xi_- \ ; & x_1+ix_2 \\ (x_1-ix_2)\xi_+ - (1-x_0)\xi_- \ ; & x_1-ix_2 \ ; & 1-x_0+\xi_+\xi_- \end{pmatrix}. \tag{4.25}$$

It is evident that these projectors are one the supertransposed of the other. They are both projectors in $\mathbb{M}_{1,2}(\mathcal{A}_{C_L})$.

5 The Connections and the Charges

Associated with any projector there is a canonical connection. Let us first consider the projector p_{-n}. The connection is given by

$$\begin{aligned} &\nabla_{-n} = p_{-n} \circ d \ : \ \Gamma^\infty(S^{2,2}, E^{(-n)}) \ \rightarrow \ \Gamma^\infty(S^{2,2}, E^{(-n)}) \otimes_{\mathcal{A}_{C_L}} \Omega^1(S^{2,2}, C_L), \\ &\nabla_{-n}(\sigma) =: \nabla_{-n}\Big(p_{-n}\,||f\rangle\rangle\Big) = p_{-n}\Big(\,||df\rangle\rangle + dp_{-n}\,||f\rangle\rangle\,\Big) \ , \end{aligned} \tag{5.1}$$

where we have used explicitly the identification $\Gamma^\infty(S^{2,2}, E^{(-n)}) = p_{-n}(\mathcal{A}_{C_L})^{2n+1}$.
The curvature $\nabla^2_{-n} : \Gamma^\infty(S^{2,2}, E^{(-n)}) \rightarrow \Gamma^\infty(S^{2,2}, E^{(-n)}) \otimes_{\mathcal{A}_{C_L}} \Omega^2(S^{2,2}, C_L)$ is found to be

$$\nabla^2_{-n} = p_{-n}(dp_{-n})^2 = |\psi_{-n}\rangle \, \langle d\psi_{-n}|d\psi_{-n}\rangle \, \langle\psi_{-n}| \ . \tag{5.2}$$

Then, by means of a matrix supertrace the first Chern class of the superbundle determined by p_{-n} is represented by the 2-superform

$$\begin{aligned} C_1(p_{-n}) \ &=: \ -\frac{1}{2\pi i}\ Str(p_{-n}(dp_{-n})^2) = -\frac{1}{2\pi i}\,\langle d\psi_{-n}|d\psi_{-n}\rangle \\ &= \ -\frac{1}{2\pi i}\Big[(dada^\diamond + dbdb^\diamond)(n - \frac{1}{4}\eta\eta^\diamond) \\ &\qquad\qquad + \frac{1}{4}(ada^\diamond + bdb^\diamond)(\eta d\eta^\diamond - \eta^\diamond d\eta) + \frac{1}{4}d\eta d\eta^\diamond\Big] \ , \\ &= \ -\frac{1}{2\pi i}\Big[n(dada^\diamond + dbdb^\diamond) + \frac{1}{4}d(a\eta^\diamond)d(\eta a^\diamond) + \frac{1}{4}d(b\eta^\diamond)d(\eta b^\diamond)\Big] \ . \end{aligned} \tag{5.3}$$

By using the coordinates on $S^{2,2}$ the previous 2-superform results in

$$\begin{aligned} C_1(p_{-n}) &= \frac{n}{4\pi}(x_0 dx_1 dx_2 + x_1 dx_2 dx_0 + x_2 dx_0 dx_1)(1 + 3\xi_-\xi_+) \\ &\quad + \frac{1}{4\pi i}\Big[(dx_1 - idx_2)\xi_+ d\xi_+ - (dx_1 + idx_2)\xi_- d\xi_- + dx_0(\xi_- d\xi_+ + \xi_+ d\xi_-) \\ &\quad + (x_1 - ix_2)d\xi_+ d\xi_+ - (x_1 + ix_2)d\xi_- d\xi_- - 2x_0 d\xi_- d\xi_+\Big] \ . \end{aligned} \tag{5.4}$$

Finally, to compute the corresponding first Chern number we need the Berezin integral over the supermanifold $S^{2,2}$. This is a rather simple task due to the fact that $S^{2,2}$ is a De Witt supermanifold over the two-dimensional sphere S^2 in $\mathbb{R}^3$. Indeed, by using the natural morphism of forms $\widetilde{\ }: \Omega^2(S^{2,2}) \to \Omega^2(S^2)$, the first Chern number yielded by the superform $C_1(p_{-n})$ is computed as [5]

$$c_1(p_{-n}) =: Ber_{S^{2,2}}\Big(C_1(p_{-n})\Big) =: \int_{S^2} \widetilde{C_1(p_{-n})} \ . \tag{5.5}$$

It is straightforward to find the projected form $\widetilde{C_1(p_{-n})} \in \Omega^2(S^2)$. The bundle projection $\Phi : S^{2,2} \to S^2$ on the body manifold S^2 is explicitly realized in terms of the body map,

$$\Phi(x_0, x_1, x_2; \xi_-, \xi_+) = (\sigma(x_0), \sigma(x_1), \sigma(x_2)) \ . \tag{5.6}$$

We recall that fermionic variables do not have body, thus they project into zero. On the other side, by denoting $\sigma_i = \sigma(x_i),\ i = 0, 1, 2$, the σ_i's are cartesian coordinates for the sphere S^2 in $\mathbb{R}^3$ and obey the condition $(\sigma_0)^2 + (\sigma_1)^2 + (\sigma_2)^2 = 1$. The projected form $\widetilde{C_1(p_{-n})}$ is found to be

$$\widetilde{C_1(p_{-n})} = \frac{n}{4\pi}(\sigma_0 d\sigma_1 d\sigma_2 + \sigma_1 d\sigma_2 d\sigma_0 + \sigma_2 d\sigma_0 d\sigma_1) = \frac{n}{4\pi} vol(S^2) \ . \tag{5.7}$$

As a consequence

$$c_1(p_{-n}) = Ber_{S^{2,2}}\Big(C_1(p_{-n})\Big) = \frac{n}{4\pi}\int_{S^2} d(vol(S^2)) = n \ . \tag{5.8}$$

It is easy to check that the supertransposed projector p_n is obtained from p_{-n} by exchanging $a \leftrightarrow a^\diamond, b \leftrightarrow b^\diamond$ and $\eta \to -\eta^\diamond,\ \eta^\diamond \to \eta$. This amounts to the exchange of coordinates $x_2 \to -x_2$ and $\xi_- \to -\xi_+,\ \xi_+ \to \xi_-$. It is than clear that the corresponding Chern class is represented by

$$C_1(p_n) = -C_1(p_{-n}) \ , \tag{5.9}$$

while the corresponding Chern number is given by,

$$c_1(p_n) = -c_1(p_{-n}) = -n \ . \tag{5.10}$$

Having different topological charges the projectors p_{-n} and p_n and the corresponding super line bundles are clearly inequivalent.

6 Graded Monopoles of any Charge

Now we are going to compute the connection 1-form associated with the canonical connection. We shall due this for the positive values of the topological charge, the construction for the negative charges being the same. Thus, given the connection (5.1), the corresponding connection 1-form on the equivariant maps,

$$A_{-n} \in End_{\mathcal{B}_{C_L}}\Big(G^\infty_{(-n)}(UOSP(1,2),C_L)\Big) \otimes_{\mathcal{B}_{C_L}} \Omega^1(UOSP(1,2),C_L) \ , \tag{6.11}$$

has a very simple expression in terms of the supervector-valued function $|\psi_{-n}\rangle$ [16, 17],

$$A_{-n} = \langle \psi_{-n} | d\psi_{-n} \rangle \ . \tag{6.12}$$

The associated covariant derivative on any $\varphi^\sigma \in G^\infty_{(-n)}(UOSP(1,2),C_L)$ is given by

$$\nabla_{-n}(\varphi^\sigma) =: \langle \psi_{-n} | \nabla_{-n}(\sigma) \rangle = \Big(d + \langle \psi_{-n} | d\psi_{-n} \rangle\Big)\varphi^\sigma \ ; \tag{6.13}$$

here $\sigma \in \Gamma^\infty(S^{2,2}, E^{(-n)})$ and we have used the isomorphism (4.13). The connection form (6.12) is anti-hermitian, a consequence of the normalization $\langle \psi_{-n} | \psi_{-n} \rangle = 1$:

$$(A_{-n})^\dagger =: \langle d\psi_{-n} | \psi_{-n} \rangle = -\langle \psi_{-n} | d\psi_{-n} \rangle = -A_{-n} \ . \tag{6.14}$$

Explicitly,

$$A_{-n} = (n - \frac{1}{4}\eta\eta^\diamond)(ada^\diamond + bdb^\diamond) + \frac{1}{8}(\eta d\eta^\diamond + \eta^\diamond d\eta) \ . \tag{6.15}$$

As for the connection 1-form A_n carrying a negative value of the topological charge one finds

$$A_n = -(n - \frac{1}{4}\eta\eta^\diamond)(ada^\diamond + bdb^\diamond) - \frac{1}{8}(\eta d\eta^\diamond + \eta^\diamond d\eta) = -A_{-n} \ . \tag{6.16}$$

The connection form A_{+1} which corresponds, we remind, to the values -1 for the topological charge, was introduced for the first time in [18] and extensively studied in [2]. Following what we did in the latter paper, we name *Grassmann (or graded) monopole* of charge $\pm n$ the connection 1-form $A_{\mp n}$.

7 The K-theory of the Supersphere $S^{2,2}$

Given a complex supervector bundle $\pi : E \to M$ over a supermanifold M one can define even and odd Chern classes [1]. Of course, the classes of both type are even cohomology classes but they get an additional graded label in $\mathbb{Z}_2$. Thus, if the bundle is of rank (r,s), so that it can be though of as having typical fiber ${C_L}^{r,s}$ and structure supergroup $GL_{r,s}(C_L)$, there are r even Chern classes $C^{(0)}_j(E) \in \check{H}^{2j}(M,\mathbb{Z})$, $j = 1,\ldots,r$, and s odd classes $C^{(1)}_k(E) \in \check{H}^{2k}(M,\mathbb{Z})$, $j = 1,\ldots,s$. Then, one proceeds to define even and odd total Chern classes and even and odd Chern characters. The two kind of classes come from the two possible projectivizations of the bundle E, an even and odd projectivization respectively.

If the bundle has rank $(1,0)$ there is only one even not trivial class $C_1^{(0)}(E) \in \check{H}^2(M,\mathbb{Z})$ and no odd classes. If the bundle has rank $(0,1)$ there is only one odd not trivial class $C_1^{(1)}(E) \in \check{H}^2(M,\mathbb{Z})$ and no even classes. Both these two classes $C_1^{(0)}(E)$ and $C_1^{(1)}(E)$ can be realized as super de Rham cohomology classes of the base supermanifold M and, at least when M is a De Witt supermanifold, they can be given a representative in terms of the curvature of a connection on the bundle, the choice of the particular connection being immaterial up to cohomologous forms.

Now, it should be clear that all analysis of this paper, especially the one in Section 4 and Section 5, can be carried over for bundles on $S^{2,2}$ of both ranks $(1,0)$ and $(0,1)$. Then, from the constructions of this paper we can conclude that the *reduced* $\widetilde{K}_0$ group of $S^{2,2}$ is the graded additive supergroup made of two copies of $\mathbb{Z}$,

$$\widetilde{K}_0(S^{2,2}) = \mathbb{Z} \oplus \mathbb{Z} \ , \tag{7.1}$$

the first copy being given an even degree while the second one gets an odd one. It is somewhat suggestive to write

$$\widetilde{K}_0(S^{2,2}) = \widetilde{K}_0(S^2) \oplus \Pi\widetilde{K}_0(S^2), \tag{7.2}$$

with S^2 the ordinary 2-dimensional sphere and Π denoting the parity change functor [20]; here $\widetilde{K}_0(S^2)$ is thought of as a $(1,0)$ supergroup so that $\Pi\widetilde{K}_0(S^2)$ is a $(0,1)$ supergroup.

From what was said at the end of Section 5 we know that by taking the supertranspose of projectors there is a change in sign in the corresponding topological charge (Chern number). Thus, supertransposing of projectors, although it is an isomorphism in 'super' K-theory is not the identity map.

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